SS 2017

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Manifolds and Group actions

Homework 11

Mandatory Exercise 1. (12 Points)
Let G be a compact n-dimensional Lie group. For $x \in G$ let L_x and R_x denote the left and right translation by x.

a) A degree k form $\omega \in \Omega^k(G)$ on G is called left invariant if $L_x^*\omega = \omega$. Show that there is a bijection between left invariant degree k forms on G and $\Lambda^k T_e^* G$. Conclude that for every $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ there exists a unique left invariant volume form ω on G with $\int_G \omega = \lambda$.
Note: $\int_G \omega$ is well defined because the Lie group G is compact!

Fix a left invariant volume form $\omega \in \Omega^n(G)$.

b) Show that for $x \in G$ the form $\eta = R_x^*\omega$ is left invariant.

c) By part a) there exists $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ such that $\eta = \lambda\omega$. Compute the constant λ (by comparing the integrals over M). Conclude that left invariant volume forms on G are also right invariant.

d) Let M be a manifold and $\varphi : G \times M \to M$ an action of G on M and $\varphi_x : M \to M$ the map $\varphi_x(p) = \varphi(x, p)$. Let g be any Riemannian metric on M. Show that

$$h_p(X_p, Y_p) = \int_G (\varphi_x^* g_p)(X_p, Y_p)\, \omega_x,$$

defines a G invariant metric on M, i.e. show that h is a metric and that $\varphi_y^* h = h$.

Mandatory Exercise 2. (8 Points)
Let $T^2 = S^1 \times S^1 \subset \mathbb{C}^2$. Consider the action $\varphi : T^2 \times \mathbb{CP}^2 \to \mathbb{CP}^2$ defined by

$$\varphi((t, s), [z_0, z_1, z_2]) = [tz_0, sz_1, z_2].$$

a) Investigate this action. i.e. determine if the action is free, effictive and/or transitive. If the action is not free, determine the fixed points and the stabilizers. If the action is not transitive determine the orbits.

Consider the following group homomorphism $f : S^1 \to S^1 \times S^1$, given by $f(s) = (s^k, s^l)$ where $k, l \in \mathbb{Z}$.

b) The Lie group homomorphism induces an action $\psi : S^1 \times \mathbb{CP}^2 \to \mathbb{CP}^2$ by

$$\psi(s, [z_0, z_1, z_2]) = \varphi(f(s), [z_0, z_1, z_2]).$$

Answer the same questions as in part a), but for this action.
Hint: This strongly depends on the number theoretic properies of k and l!

Suggested Exercise 1. (0 Points)
Show that a G action on M is proper, if and only if for any two compact subsets $K_1, K_2 \subset M$ the set

$$\{g \in G \,|\, (g \cdot K_1) \cap K_2\}$$

is compact.

Hand in on 10th of July in the pigeonhole on the third floor.